Fourier Diagonalization of Circular Matrices

Circular matrices creep into a lot of problems where one has interactions between similar elements in a system. The interaction matrix is usually only a function of the relative difference between the elements. Because these matrices are so common, it is good to find analytical tools which make dealing with them easier. Here we present a method whereby one can diagonalize a circular matrix in Fourier space, making easier analysis possible.

We start with a definition of a Fourier transformation matrix
\[
U_{jk} \equiv \frac{1}{\sqrt{N}} e^{i2\pi jk/N}
\]
and note the following theorem

**Theorem 1** \(U\) is a unitary transformation matrix.

**Proof**

\[
(U^\dagger U)_{\alpha\beta} = \frac{1}{N} \sum_{j,k=1}^{N} e^{-i2\pi j\alpha/N} e^{i2\pi j\beta/N} = \frac{1}{N} \sum_{j=1}^{N} e^{i2\pi j(\beta-\alpha)/N}
\]

define: \(y \equiv e^{i2\pi j(\beta-\alpha)/N}\)

\[
y^N = \left(e^{i2\pi j(\beta-\alpha)/N}\right)^N = e^{i2\pi j(\beta-\alpha)} = 1 \quad \beta - \alpha \text{ is an integer}
\]

\[
(U^\dagger U)_{\alpha\beta} = \frac{1}{N} \sum_{j,k=1}^{N} y^j = \frac{y}{N}(1 + y + y^2 + \cdots + y^{N-1})
\]

\[
= \begin{cases}
\frac{N}{N} \left( \frac{1}{1-y} - y^N \frac{1}{1-y} \right) & \text{for } |y| < 1 \quad (\alpha \neq \beta) \\
\frac{1}{N} \frac{y^N}{1-y} & \text{for } |y| = 1 \quad (\alpha = \beta)
\end{cases}
\]

\[
= \begin{cases}
0 & \text{for } \alpha \neq \beta \\
1 & \text{for } \alpha = \beta
\end{cases}
\]

\[
(U^\dagger U)_{\alpha\beta} = \delta_{\alpha\beta}
\]

Therefore, \(U\) is a unitary transformation matrix. \(\square\)

Now we define a circular matrix \(M\) as a matrix which is only a function of the relative difference of its indices.

define: circular matrix \(M_{jk} = M_{jk}(|j - k|)\)

\[
= M(j - k) = M(j - k \pm N) \quad (j, k = 1, \ldots, N)
\]

Next we transform the problem into Fourier space by using the transformation matrix \(U\).

define: \(D \equiv U^\dagger MU\)

\[
D_{\alpha\beta} = \frac{1}{N} \sum_{j,k=1}^{N} e^{-i2\pi j\alpha/N} M_{jk} e^{i2\pi k\beta/N}
\]

\[
= \frac{1}{N} \sum_{j,k=1}^{N} M_{jk} e^{-i2\pi(j\alpha - k\beta)/N} = \frac{1}{N} \sum_{j,k=1}^{N} M(j - k) e^{-i2\pi(j-k)\alpha/N - i2\pi k(\alpha - \beta)/N}
\]

\[
= \frac{1}{N} \sum_{k=1}^{N} e^{-i2\pi k(\alpha - \beta)/N} \sum_{j=1}^{N} M(j - k) e^{-i2\pi(j-k)\alpha/N}
\]
It is now time to propose a theorem which will simplify the expression for $D$.

**Theorem 2** The expression $\sum_{j=1}^{N} M(j - k)e^{-i2\pi(j-k)\alpha/N}$ is independent of $k$.

**Proof**

Define: $l \equiv j - k$ ($k = 1, 2, \ldots, N$ \quad $l = 1 - k, 2 - k, \ldots, N - k$)

$$\left( \sum_{j,k=1}^{N} \right) = \left( \sum_{k=1}^{N-k} \sum_{l=1-k}^{N-k} \right)$$

$$\sum_{j=1}^{N} M(j - k)e^{-i2\pi(j-k)\alpha/N} = \sum_{l=1-k}^{N-k} M(l)e^{-i2\pi\alpha l/N}$$

$$= \sum_{l=1-k}^{0} M(l)e^{-i2\pi\alpha l/N} + \sum_{l=1}^{N-k} M(l)e^{-i2\pi\alpha l/N}$$

Circular matrix assumption $\Rightarrow$

$$= \sum_{l=N-k+1}^{N} M(l)e^{-i2\pi\alpha l/N} e^{-i2\pi\alpha} + \sum_{l=1}^{N-k} M(l)e^{-i2\pi\alpha l/N}$$

$$= (e^{-i2\pi\alpha} = 1) \quad \Rightarrow \quad \sum_{l=1}^{N} M(l)e^{-i2\pi\alpha l/N}$$

Therefore,

$$\sum_{j=1}^{N} M(j - k)e^{-i2\pi(j-k)\alpha/N} = \sum_{l=1}^{N} M(l)e^{-i2\pi\alpha l/N}$$

is independent of $k$. \hspace{1cm} \Box

Finally we make the convenient definition

Define: $\lambda_\alpha \equiv \sum_{l=1}^{N} M(l)e^{-i2\pi\alpha l/N}$

which then simplifies $D$ to

$$D_{\alpha\beta} = \lambda_\alpha \frac{1}{N} \sum_{k=1}^{N} e^{-i2\pi k(\alpha-\beta)/N}$$

$$= \lambda_\alpha (U^\dagger U)_{\alpha\beta}$$

$$= \lambda_\alpha \delta_{\alpha\beta}$$